Estimating the area and number of bar crossings between refiner plates
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KEYWORDS: LC Refiner, Refiner Plates, Area of intersection, Perimeter of intersection, Number of bar crossings

SUMMARY: A novel methodology for estimating the area and perimeter of intersection between two or more intersecting surfaces, without having to determine or order the points of intersection, is presented. No restriction is placed on complexity of the intersecting geometry except that it must represent a closed region. This method is well-suited for cases in which the area and perimeter of intersection are complicated, are not strongly peaked in a very small region, and when relatively low accuracy is tolerable. In this study, we examine the time-dependent evolution of the area and number of bar crossings for 320 different low consistency refiner plate configurations with similar geometrical patterns for stator plate and rotor plate using this methodology. Empirical correlations are presented to relate the plate parameters to bar intersection area. Finally, we interpret our findings in terms of the industrially accepted parameters used to characterize the action of refiner plates.

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In this work, we present a numerical method to estimate the area, and perimeter, of intersection between two refiner plates. Although the methodology is general in nature and can be applied to any style or combination of refiner plates, the motivation of the present study stems from one application, namely, the low consistency (LC) refining of mechanical pulps. One of the open remaining questions in this area is the interrelationship between the plate pattern, energy consumption, and the changes in fibre morphology. Understanding how plate design affects this relationship is difficult, as the number of contact points between these plates is vast and the local area of intersections between the plates form a system of closed-polygons whose shape evolve and translate as a function of time. This problem has yet to be attempted numerically.

Before proceeding to discuss our numerical method, it is instructive to first briefly review the action of refiners; an excellent review of this topic is given by Page (1989). LC refiners are mechanical devices employed to modify the papermaking fibre morphology. They are rotary devices having bar patterns machined onto a rotor and a stator, see Fig 1. The rotor and stator are disc or conical in shape and are separated by a gap of about 0.1 to 0.5 mm. Although this work is presented for disc refiner plates, the methodology proposed can be used for any shape of refiners. Papermaking fibres flow in the grooves of the plate and are trapped between opposing bars in the narrow gap between the rotor and stator. The fibres are “beaten” or “refined” by repeated impacts between the bars. While much is known about the effects of refining on paper strength, relatively little is known on how it is carried out. Clearly, force is imposed upon the fibres, but the magnitude of the force applied to the individual fibres has yet to be determined.

Despite a lack of fundamental knowledge of the force applied to the fibres in the refiners, semi-empirical models have been developed to describe the action of refiners. These models date back as far as the early 20th century (Kirchner (1906), Pfarr (1907)). Since then, a number of groups have proposed characterization schemes, see for e.g. Lewis & Danforth (1962), Leider & Nissan (1977), Danforth (1969), and Kerekes (1990), and suggest that the refining action is governed by a number of different parameters with the intensity of each impact $I$, key to understanding the changes in fibre morphology. The intensity of impact is related to variables such as the net power expended upon the suspension $P_{net}$, the rotational frequency $N$, and on the plate pattern. It is widely accepted that the plate pattern is the key to understanding the changes in morphology.

One of the earliest and most widely adopted estimates for $I$ was popularized by Brecht (1967) who proposed that the intensity of impact was related to the pattern of opposing bars. He characterized the bar pattern as the...
effective edge length per revolution, \( L_r \) \( \left( \frac{m}{rev} \right) \), using the equation
\[
L_r = n_r n_s L
\]
and defined the intensity of impact as:
\[
I = \frac{P_{\text{net}}}{N L_r}
\]
where \( n_r \) is the number of rotor bars, \( n_s \) is the number of stator bars, \( L \) is the effective bar length (m), and \( P_{\text{net}} \) is the net refining power applied to fibres by the refiner (kW). The intensity of treatment was later related to the plate pattern by replacing \( L_r \) with \( BEL \) using (TAPPI (1971)).
\[
I = \frac{P_{\text{net}}}{N \cdot \text{BEL}}
\]
where \( \text{BEL} \) is a standardized measure in the industry and is estimated using TAPPI standard TIP 0508-05 (1994):
\[
\text{BEL} = \frac{4\pi^2}{3} \left( \frac{R_0^2 - R_1^2}{\cos a (B+G)^2} \right)
\]
where \( R_0 \) and \( R_1 \) are the length of bars on rotor and stator respectively, while Kerekes (1990) considered
\[
I \propto \frac{1}{X_1 X_2}
\]
where \( X_1 \) and \( X_2 \) are the length of bars on rotor and stator respectively, while Kerekes (1990) considered
\[
I \propto \frac{1}{(B+G)^3 (1+2\tan a)}
\]
These equations have proved useful in practice, but have shortcomings from a theoretical standpoint. For example, the functional dependency between \( I \) and the plate parameters \( B \) and \( G \) vary from the squared relationship proposed by Brecht (1967) to a cubic one as advanced by Kerekes (1990). Similarly, Danforth (1969) considers that the intensity to be related to the total length of the bars and not to the length of intersection between them.

In addition to this debate, there are some authors, i.e. Kline (1978), who suggest that \( I \) is related to the effective refining area rather than the length of the intersecting area. Clearly, there is very little agreement on the form of \( I \) in the literature.

To address this situation, Roux and his co-workers (2009) recently proposed a new concept for assessing the intensity of refining by using a physical analogy between a slice of a beater and an annulus of a disc. They demonstrate a relationship between changes in pulp properties, such as fibre length and freeness, to intensity defined as:
\[
I = \frac{P_{\text{net}}}{N} \left( \frac{3\beta^2}{16 \pi^2 \sin(2\alpha+\beta) \sin\left(\frac{\beta}{2}\right)} \right) \left( \frac{(B+G)^2}{R_0^2 - R_1^2} \right)
\]
In their work, they neglect the temporal variation of quantities such as bar angle and use average quantities in the first order as representative values. This estimate is quite reasonable to predict fibre length reduction and \(^{13}SR\) evolution on fibres. Beyond this work, the deviations from the average of the local variables have not been estimated numerically. Here, we advance a numerical scheme to characterize the time-dependent intersecting area and number of bar crossing points formed between the rotor and stator plates during refining. We feel that the time-dependent variations in the patterns formed may lead to insight into the beating effect and the definition of intensity.

In this paper we characterize the instantaneous variations in the area and perimeter formed between two refiner disks numerically. We highlight the numerical method by illustrating the methodology using a simpler toy problem. This methodology is extended to the more industrially relevant case of two disc refiner plates. The time-dependent variations are characterized for approximately 320 different low consistency refiner plates. Finally, we interpret our findings in terms of the current estimates found in the literature.

### The Toy Problem

Computing the area and perimeter of intersecting objects is one of the most fundamental aspects of computational geometry. As the region of intersection forms a polygon itself, its area and perimeter can be calculated once the points of intersection are determined and the vertices are ordered so that a simple path can be formed. The degree of complexity of this problem varies if the polygon is concave or convex. Convex polygons are those in which every internal is less than 180°; a polygon that is not convex is concave. The convex problem is considered the simpler of the two cases as the ordering of the vertices is relatively straightforward. Shamos & Hoey (1976), O’Rourke et al. (1982) and Saab (1997) present efficient methods for doing so. With concave polygons, Chazelle & Dobkin (1985) demonstrate a methodology of decomposition to form a system of convex polygons. In this work we present a methodology that avoids the computationally intensive steps of ordering the vertices and detecting the intersecting points between intersecting polygons. We advance a methodology, which (somewhat) resembles a Monte-Carlo approach.

Before we highlight our method through a simple example we must formally define the terms used. Let \( D = \{D_1, D_2, D_3, \ldots\} \) be collection of polygons in \( R^2 \) that intersect and let \( \text{int}(D_1) \) and \( \text{ext}(D_1) \) denote the points of \( R^2 - D_1 \) that are interior and exterior, respectively, to \( D_1 \). To specify particular regions within \( D \), we define \( D_1 \cap D_2 \) to represent \( \text{int}(D_1) \cap \text{int}(D_2) \). Further, we define \( F(x, y) = \{f_1, f_2, f_3, \ldots\} \) to be a collection of functions evaluated over discrete set of points, \( (x_j, y_k) \) where \( j, k = 1, 2, 3 \ldots M \), using the following functional form:

\[
f_i = \begin{cases} 1 & \text{int}(D_i) \\ 0 & \text{otherwise} \end{cases}
\]

To highlight this, we consider the intersection of the polygons \( D_1 \) and \( D_2 \) as shown in Fig 2(a). We create two separate functions \( f_1 \) and \( f_2 \) through use of Eq. 8 and visualize the domain by summing these functions to create one image (Fig 2(b)). Here we see the common intersecting region \( D_1 \cap D_2 \) is given by the region in which the function is equal to two. Once the image of the region of interest is created, the area \( A \) and perimeter \( P \) can be deduced through standard image analysis techniques.
Fig 2. (a) The geometry of two intersecting squares $D_1$ and $D_2$ contained in a larger rectangle $D_3$. $D_1$ and $D_2$ are squares with an edge length of $l_1$ and $l_2$ and are centered at $(0,0)$ and $(d,0)$, respectively. (b) An image of each polygon in (a) is defined through use of Equation 8. This image is created by summing all the functions at each grid point.

Fig 4. (a) A representative image of a stator plate with the cluster angle of $\beta$. The value of 1 represents the surface of the bars. (b) A representative image of the rotor in which the angular velocity of the plate $\omega$, measured relative to the horizontal is defined. Here, $t$ represents time. (c) The summation of the images given in (a) and (b).

To highlight the utility of this method we report the error in determining the area of $D_1 \cap D_2$ as a function of number of nodes in the $x$ and $y$ directions ($M$) in Fig 3. Here we see that the relative error diminishes somewhat linearly with increasing grid size. A relatively crude estimate, i.e. a relative error of $10^{-3.5}$ can be achieved with approximately 1000 grid points in each direction; this is a relatively low-accuracy result. Higher accuracy results can be achieved with higher-order segmentation algorithms to determine the edges of the object.

There are a number of benefits to this numerical methodology as opposed to using the exact solution. To begin, this method produces reasonable estimates for cases in which the area and perimeter of intersection are not strongly peaked in a very small region. This methodology is general and can be applied for any shape of closed bodies that can be represented as a polygon.

To highlight this we present another toy problem in Appendix A to demonstrate this. Finally, we realize the true benefit of this approach when we have a vast number of intersecting objects in the image. With the exact method, i.e. determining the vertices and then ordering the problem, the computational intensity increases with the number of intersecting objects. The true benefit of this method is when we extend the problem to estimating the area and perimeter of thousands of simultaneously intersecting bodies. With this methodology, the computational requirements do not increase as rapidly with increasing number of intersecting objects.

Results

At this point we turn our attention to reporting on the algorithm for the industrially relevant case of two overlapping disc refiner plates, see Fig 4. Images of the stator and rotor are represented in the first two panels of this figure. The angular speed of the rotor is defined by $\omega$ and its angular position at any time $t$ is defined by $\omega t$. We used Eq 8 to generate the images. As with the toy problem, we highlight the region of interest by adding the functions which represent the rotor and stator. In Fig 4(c), the regions which are assigned the value of two represent the intersecting regions between opposing bars of the rotor and stator.

It should be noted that the case illustrated in this figure is considered as a very coarse bar pattern. We use this pattern for clarity in the presentation. Industrially relevant patterns contain approximately 30 times more bars on each rotor and stator. Four snapshots of the intersection patterns at different positions of the rotor are given in Fig 5. We see that the most common object of intersection formed is a parallelogram with equal adjacent length (lozenge). The unique property of such a parallelogram is that its area is proportional to the square of its perimeter.

In total 320 simulations were conducted in which $[B, G, \alpha, \beta, R_0] \in [1.3 mm, 3.2 mm] \times [2.4 mm, 4 mm] \times [7.5^\circ, 17.5^\circ] \times \omega [7.5^\circ, 17.5^\circ] \times [0.2159 m, 0.4318 m]$. The sensitivity of the solution is shown in Fig 6 as a function of the number of grid points in the $x$ and $y$ directions for one time step. Here, we see that although the estimated value of $A$ generally diminishes with increasing $M$, we find that the solution varies by less than 2% for cases with $M > 1000$. All simulations were conducted with $M = 3000$. 

Fig 5. A schematic of the complex intersection pattern created when the rotor passes over the stator plate. Four images are given at different times: (a) $\omega t / \beta = 0.25$; (b) $\omega t / \beta = 0.5$; (c) $\omega t / \beta = 0.75$; and, (d) $\omega t / \beta = 1$.

Fig 7. The variation of area $A$, as a function of rotational position for a number of different plate configurations. In (a) the bar angle $\alpha$ is held constant at a value of $7.5^\circ$. In (b) the value of $\alpha$ is set to $7.5^\circ$, $0.4318m$ and $0.2669m$ respectively. The roman numeral in these figures refer to various bar and groove widths and are defined in table 1. The average value as well as the standard deviation of all 8 cases is summarized in the Table 1.

Fig 6. An estimate of the sensitivity of the area $A$ as a function of number of grid points $M$, in the x and y directions. For this simulation $B = 3.2 \text{ mm}$, $G = 2.4 \text{ mm}$, $\beta = 7.5^\circ$, $\alpha = 7.5^\circ$, $R_o = 0.4318 \text{ m}$, and $R_i = 0.2669 \text{ m}$.

Fig 8. The relationship between the normalized average value of instantaneous area of bar crossings and its standard deviation with the cluster angle of $\beta = 7.5^\circ$. The results shown are for the cases with $M = 3000$.

At this point we turn our attention to the main findings of this work and examine the evolution of area with rotational position. In Fig 7, eight simulations are presented with the conditions given in Table 1. The first observation that can be made from this figure is that there are two dominant frequencies comprising the signal. The low frequency signal has a period of the cluster angle $\beta$. The high frequency signal has a wave length of approximately $\lambda$. The second observation is that with decreasing bar size (i and iii or ii and iv) the area of intersection diminishes. The third observation that can be made is that the average value of $A$ in each one of these traces is related to the ratio of $B/(B+G)$. Finally, quantitatively we observe that the variance in each of these signals diminishes with the magnitude of the area. This is quantified in Fig 8, in which the average area, normalized by the area of the plate, defined as:

$$A^* = \frac{1}{\pi (R_o^2 - R_i^2)} \frac{\omega}{2\pi} \int_0^{2\pi} A dt$$

is shown as a function of the standard deviation ($\sigma^*$) of the signal.

We also notice that for a plate with smaller bar size, more homogeneity in the area of bar crossings is expected. In order to prove this, we have compared 2 different configurations of plates in the Table 1 (i.e. (i) and (iii)) in which we have a coarser distribution of bar patterns in (i) with $B = 3.2 \text{ mm}$ comparing to (iii) with $B = 1.3 \text{ mm}$. These 2 plates have similar groove width of $G = 2.4 \text{ mm}$, bar angle of $\alpha = 7.5^\circ$ and cluster angle of $\beta = 7.5^\circ$. Fig 9 shows this comparison.
Table 1. 8 cases studied in Fig 7 with various bar width, groove width and bar angle

<table>
<thead>
<tr>
<th></th>
<th>α = 7.5°</th>
<th></th>
<th>α = 15°</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(i)</td>
<td>(ii)</td>
<td>(iii)</td>
</tr>
<tr>
<td>B (mm)</td>
<td>3.2</td>
<td>3.2</td>
<td>1.3</td>
</tr>
<tr>
<td>G (mm)</td>
<td>2.4</td>
<td>4</td>
<td>2.4</td>
</tr>
<tr>
<td>$B + G$</td>
<td>0.57</td>
<td>0.44</td>
<td>0.35</td>
</tr>
<tr>
<td>$\Lambda^*$</td>
<td>0.214</td>
<td>0.138</td>
<td>0.084</td>
</tr>
<tr>
<td>$\sigma^*$</td>
<td>0.013</td>
<td>0.008</td>
<td>0.005</td>
</tr>
<tr>
<td>$N_c$ (contact)</td>
<td>3817</td>
<td>2417</td>
<td>7818</td>
</tr>
<tr>
<td>$\sigma^*$</td>
<td>342</td>
<td>213</td>
<td>636</td>
</tr>
</tbody>
</table>

Although the magnitude of area of bar crossings is smaller for the plate with smaller width, a more uniform distribution in the area of bar crossings is achieved.

Fig 9. The probability distribution of the area of bar crossings for the 2 different plate configurations (i.e. (i) and (iii)) reported in the table 1. In (a), $B = 3.2\text{mm}$, $G = 2.4\text{mm}$, $\alpha = 7.5^\circ$, $\beta = 7.5^\circ$ and $Ro = 0.4318\text{m}$. In (b), $B = 1.3\text{mm}$, $G = 2.4\text{mm}$, $\alpha = 7.5^\circ$, $\beta = 7.5^\circ$ and $Ro = 0.4318\text{m}$.

At this point, we turn our attention to developing a rule of thumb regarding the relationship between $A^*$ to the plate parameters. We do so empirically and attempt to fit an equation of the form:

$$A^* = 1.14\left(\frac{B}{B + G}\right)^2 \left(\frac{\beta}{2\pi}\right)^{0.4} \left(\frac{1}{\sin\alpha}\right)^{0.5} \quad [10]$$

We also highlight the importance of variations in the distribution of instantaneous area in Fig 8. By changing a plate configuration in a way which a larger area of bar crossings is provided, a higher variation in the magnitude of bar crossings is expected. This result is in accordance with findings of Brecht (1965) where he finds a higher noise level in a smaller bar angles. This might be a result of the higher variation in the area distribution of bar crossings at smaller angles.

Fig 10. (a) The relationship between $A^*$ and the dimensionless group $\frac{B}{B + G}$. For clarity, this relationship is shown at two different bar angles $\alpha$ and two different cluster angles $\beta$. The results shown are for the cases with $M = 3000$, $Ro = 0.4318\text{m}$, and $Ro = 0.2159\text{m}$. The roman numeral in this figure refers to various bar and cluster angles and are: (i): $\alpha = 7.5^\circ$ and $\beta = 15^\circ$ (ii): $\alpha = 7.5^\circ$ and $\beta = 7.5^\circ$, (iii): $\alpha = 17.5^\circ$ and $\beta = 15^\circ$, (iv): $\alpha = 17.5^\circ$ and $\beta = 7.5^\circ$. In (b), we display the goodness of the fit between the numerical results (abscissa) and the predictions using Eq 10.

In Fig 10(a), we highlight the utility of the correlation by plotting $A^*$ as a function of $B/(B + G)$ for two $\alpha$ and two different $\beta$. The utility of the fit for all cases was tested and is shown in Fig 10(b). Excellent agreement is found as the correlation coefficient $r^2$ was determined to be 0.99.

In the final part of this section we examine the length of the perimeter formed during intersection. As mentioned previously, most polygons formed in the domain are
Fig 11. The relationship between the dimensionless perimeter $P^*$ and area for all cases simulated. $P^*$ represents the ratio of the sum of the perimeters to the total length of the edges of the bars.

Discussion

Another outcome of the current work is the possibility of detecting the number of bar crossings ($N_{\text{crossings}}$) at each rotational position which seems to play a major role in the intensity of refining. To make this clear, In Fig 4(c) each segment has 7 number of bar crossings which leads to a total of $56$ bar crossings at $\omega t = 0$ for the case shown. As the location of stator and rotor bars vary both temporarily and spatially along the radius of the refiner plate, a different number of bar crossings are created. Fig 12(a) shows the total number of bar crossings for a plate with $B = 2 \text{mm}$, $G = 3.2 \text{mm}$, $\alpha = 15^\circ$, $\beta = 7.5^\circ$ and $R_o = 0.4318 \text{m}$.

The average number of bar crossings is also shown to be a function of our distinguished dimensionless group of $B/(B + G)$ as well as the bar angle $\alpha$. In Fig 12(b) a correlation between $N_c$, $B/(B + G)$ and $\alpha$ has been shown for a constant plate size of $R_o = 0.4318 \text{m}$ and $\beta = 7.5^\circ$. Different slopes on the graph represent different bar widths to be (i): $B = 1.3 \text{mm}$, (ii): $B = 1.6 \text{mm}$, (iii): $B = 2 \text{mm}$ and (iv): $B = 3.2 \text{mm}$.

One of the main findings of this graph is that at a constant $B$, increasing $G$ will decrease the average number of bar crossings while decreasing $B$ at a constant $G$ will increase the number of crossings in one revolution of rotor disc in front of the stator disc. This can also be seen in the table of Fig 7 where decreasing bar width from $B = 3.2 \text{mm}$ to $B = 1.3 \text{mm}$ at a constant groove width of $G = 2.4 \text{mm}$ and bar angle of $\alpha = 7.5^\circ$, (i.e. (i) to (iii)) will decrease the averaged normalized area of bar crossings from $A' = 0.214$ to $A' = 0.084$. It should be noted that in this case, the standard deviation of normalized area will decrease as well while the average number of bar crossings increases from $N_c = 3754$ contact/rev to $N_c = 7488$ contact/rev. Here, $N_c$ is the average number of bar crossings over the time domain, i.e.

\[
N_c = \frac{\omega}{2\pi} \int_0^{2\pi} N_{\text{crossings}}(t)dt
\]  

\[
N_c = \frac{\pi(R_o^2 - R_i^2)}{(B + G)^2} \sin(2\alpha + \beta)
\]

With our methodology, we too can estimate the average number of crossings and compare this to Roux’s results, see Fig 13. A subset of the total number of simulations are presented in this figure in order to highlight the trend. What we find is that there is a linear relationship between these two studies when $B$ and $G$ are varied and all other parameters are held constant. Discrepancies between these works occur when bar angle is varied. The reason for this is not clear. In general though we find that our
Fig 14. (a) The relationship between the sum of the perimeters $P^*$ and the \( \text{BEL} \) defined using Eq 4. (b) The relationship between \( N_c \) and the \( \text{BEL} \). This relationship is shown at cluster angle \( \beta = 7.5^\circ \), \( R_o = 0.4318 \) m and \( M = 3000 \). The methodology gives the same order of magnitude, but consistently lower value of \( N_c \) than that of Roux.

In addition to this, we attempt to interpret \( \text{BEL} \), as defined by Eq 4, in terms of the output of our code. The natural starting point for this comparison is \( P^* \). We do however find that \( \text{BEL} \) is proportional to \( \frac{N_c}{\cos^2(\beta)} \), as shown in Fig 14(b). These results may imply that \( \text{BEL} \) is not a very good predictor of intersecting length but is an excellent representation of the number of crossing points; a parameter advanced by Roux as important in understanding the relationship between plate parameters to the beating effects.

Conclusion

A novel methodology is presented to characterize the patterns formed through the intersection of grinding surfaces. The methodology is fast and robust and is useful for cases in which a low-order accuracy result is tolerable. The benefit of the methodology is that no ordering of the vertices, formed in the region of intersection, is required. The utility of the code was tested for the industrially relevant case of low consistency refiners. We simulated the patterns formed between 320 different plates and characterized the area, perimeter and number of crossing points. An empirical relationship was advanced between the plate parameters and the average area of intersection. We demonstrate that the length of the perimeter formed from the intersecting bodies is proportional to the square root of its area. We also showed the relationship between the average area of intersection and its variations.

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Appendix
Two Intersecting Circles
The method that is introduced in this work is not just limited to simple polygons. Since any curved edge can be created by infinite number of straight lines, we expect our method to work for curved objects as well if we represent that as a high-order polynomial. Intersecting two circles is one example of curved overlapping objects. Here, for simplicity, we consider two intersecting circles centered at (0, 0) and (d, 0), formed by polygons with a large number of vertices, (6281 vertices). These are labeled as $C_1$ and $C_2$ in Fig A.1. The exact value of the area of intersection was determined using the analytical solution given by Chow & Ruskey (2003). With $M = 2000$ we estimate the area to be $0.307 m^2$ which is accurate to 5 significant figures. The relative error in this case is $1.7 \times 10^{-2}$.

![Diagram of two intersecting circles](image)

Fig A.1. The area of intersection between two intersecting circular bodies consist of 6281 vertices and with $M = 2000$. The radius of each circle is 0.5m.